Perfect Numbers

A perfect number N is one which is equal to the sum of all its divisors (including 1 but excluding N). It is sometime better to include the number itself as well, in which case, the sum of all its divisors is equal to $2 \times N$.

In what follows, p, q, r etc. are prime numbers. N is an integer. N_s is the sum of the divisors of N excluding N. N_{s+} is the sum of the divisors including N. In general it is better to consider N_{s+} which is therefore equal to $2N$

Various cases are considered.

N = *p*ⁿ

$$\begin{gathered} N_{s+} = 1 + p + p^2 + \ldots + p^n = \frac{p^{n+1} - 1}{p - 1} = 2p^n \\ p^{n+1} - 1 = 2p^n(p - 1) \\ p^n(2 - p) = 1 \end{gathered} \tag{1}$$

The only solution to this equation is $p = 1$, $n = 0$. There are therefore no perfect numbers of the form p^n (other than 1 which is not usually counted as a perfect number.)

$N = 2p^n$

$$\begin{gathered} N_{s+} = 1 + p + p^2 + \ldots + p^n + 2(1 + p + p^2 + \ldots + p^n) \\ = 3\frac{p^{n+1} - 1}{p - 1} = 4p^n \\ 3(p^n - 1) = 4p^n(p - 1) \\ p^n(4 - p) = 3 \end{gathered} \tag{2}$$

The only solution to this is $p = 3$. $n = 1$. This gives us $N = 6$ our first perfect number.

$N = 3p^n$

$$\begin{gathered} N_{s+} = 1 + p + p^2 + \ldots + p^n + 3(1 + p + p^2 + \ldots + p^n) \\ = 4\frac{p^{n+1} - 1}{p - 1} = 6p^n \\ 4(p^n - 1) = 6p^n(p - 1) \\ p^n(6 - 2p) = 4 \end{gathered} \tag{3}$$

The only solution to this is $p = 2$. $n = 1$. This gives us $N = 6$ as before.

$N = pq$

$$\begin{gathered} N_s = 1 + p + q = pq \\ 1 + p = q(p - 1) \\ q = \frac{p + 1}{p - 1} \end{gathered} \tag{4}$$

If $p + 1$ is divisible by $p - 1$ then 2 (the difference between them) must be divisible by $p - 1$.

The only numbers that divide into 2 are 2 and 1. This gives the solutions $p = 3$ and $p = 2$ respectively. This implies that $q = 2$ and $q = 3$. In both cases $N = 6$.

$N = 2^n p$

$$1 + 2 + 2^2 + \ldots + 2^n + p + 2p + 2^2 p + \ldots + 2^n p = 2 \times 2^n p$$
$$(1 + 2 + 2^2 + \ldots + 2^n)(1 + p) = 2 \times 2^n p$$
$$(2^{n+1} - 1)(1 + p) = 2^{n+1} p \tag{5}$$
$$2^{n+1} + 2^{n+1} p - 1 - p = 2^{n+1} p$$
$$p = 2^{n+1} - 1$$

Primes with the form $2^{n+1} - 1$ are called Mersenne primes. 51 Mersenne primes are known and each of these has a corresponding perfect number. It is well known that the exponent of a Mersenne prime must itself be prime hence

$$p = 2^{n+1} - 1 = M \quad \text{(where } n + 1 \text{ is prime)} \tag{6}$$

$$N = 2^n M = \frac{(1 + M).M}{2} \tag{7}$$

It is worth noting that the value $n = 1$ leads to $N = 6$, as before; $n = 2$ leads to $N = 28$ etc.

$N = p^n q \quad (p > 2)$

$$\frac{p^{n+1} - 1}{p - 1}(1 + q) = 2 p^n q$$
$$p^{n+1} - 1 + p^{n+1} q - q = 2 p^{n+1} q - 2 p^n q \tag{8}$$
$$p^{n+1} - 1 = q(p^{n+1} - 2 p^n + 1)$$

Since q must be at least 2, $p^{n+1} - 1$ be at least $2 \times (p^{n+1} - 2 p^n + 1)$.

$$p^{n+1} - 1 \geq 2 p^{n+1} - 4 p^n + 2$$
$$p^n (4 - p) \geq 3 \tag{9}$$

so the only possibility is $p = 3$, $n = 1$ in which case $q = 2$ and $N = 6$.

$N = p^n q^m \quad (p > 2, n > 1, q > p, m > 1)$

$$\frac{p^{n+1} - 1}{p - 1} \frac{q^{m+1} - 1}{q - 1} = 2 p^n q^m \tag{10}$$

As p and q are increased the fractions on the LHS get smaller and in the limit

$$\frac{p^{n+1} - 1}{p - 1} \frac{q^{m+1} - 1}{q - 1} \rightarrow p^n q^m \tag{11}$$

from above. This suggests that we should consider only cases where p and q are small. As soon as we reach a point where the LHS is smaller than the RHS (i.e. when there is a deficit), further increases in p and q will only make matters worse.

We know that when $p = 2$ and $q = 3$ then $N = 2 \times 3$ is a solution. We also know that $2 \times 3^2 = 18$ and $3 \times 2^2 = 12$ are not so the smallest new case is $2^2 \times 3^2 = 36$. This number is excessive (i.e. the sum of its factors is greater than N). It is instructive to see what happens to $N = 2^n 3^m$ as n and m *are* increased by calculating the percentage excess (i.e. $(N_s - N)/N \times 100$.

It turns out that all numbers of the form $2^n 3^m$ are excessive and, as far as I can tell, that the percentage excess gets larger and larger as n and m increase.

The case of $p = 2$ and $q = 5$ is more interesting. Here is a table of results:

	5^1	5^2	5^3	5^4	5^5
2^1	– 10	– 7	– 6.4	– 6.2	– 6.2
2^2	+ 5	+ 8.5	+ 9.2	+ 9.3	+ 9.3
3^3	+ 12.5	+ 16.2	+ 17	+ 17.2	+ 17.1
4^4	+ 16.2	+ 20.1	+ 20.9	+ 21	+ 21
5^5	+ 18.1	+ 22	+ 22.8	+ 23	+ 23

It is clear that, apart from the first row, these numbers are all excessive. The only possibility of a perfect number is in the first row – but we have already proved that there are no perfect numbers of the form $2p^n$ (other than 6).

The next case to consider is $p = 3$ and $q = 5$ but it appears that all these are deficient. I conclude that it is highly probable that no perfect numbers of the form $p^n q^m$ exist.

If other perfect numbers exist, they must consist of at least three primes.

N = 2*pq* (*q* > *p*)

$$\begin{gathered} 1+2+p+q+2p+2q+pq=2pq \\ 3+3p+3q=pq \\ 3(1+p+q)=pq \end{gathered} \tag{12}$$

This means that either p or q must be equal to 3. If $p = 3$ then:

$$\begin{gathered} 3(1+3+q)=3q \\ q=2 \end{gathered} \tag{13}$$

This is impossible because q is supposed to be a prime greater than 3.

N = 2*ⁿpq* (*n* > 0, *q* > *p*)

$$\begin{gathered} (1+2+2^2+\ldots+2^n)(1+p+q+pq)=2\times 2^n pq \\ (2^{n+1}-1)(1+p+q+pq)=2^{n+1}pq \\ (2^{n+1}-1)(1+p+q)+(2^{n+1}-1)pq=2^{n+1}pq \\ (2^{n+1}-1)(1+p+q)=pq \end{gathered} \tag{14}$$

This implies that $(1 + p + q)$ must be equal to either p or q. But this is impossible since it is obviously larger than both.

Similar arguments show that all numbers of the form $2^n pqrs$... i.e. all even numbers cannot be perfect.

N = *pqr* (*p*, *q* & *r* > 2)

$$1+p+q+r+pq+qr+pr=pqr \tag{15}$$

On the face of it, this is perfectly possible. The sum of 7 odd numbers is an odd number but if we set $p = q = r = a$ then:

$$1+3a+3a^2=a^3 \tag{16}$$

The solution to this equation is about 3.85. Roughly speaking, if p, q and r are smaller than this then the quadratic expression will win and if they are larger then the cubic term will win. We only need to see what happens to a few small values of p, q and r to get an idea of what happens.

With $p = 7$, $q = 5$ and $r = 3$, $N = 105$ and $N_s = 96$ which is a deficiency of 17%.

As we increase the values of the primes, the deficiency gets worse so there are no perfect numbers of this form either.

$N = p^n q^m r^l$ (p, q & r > 2)

$$\frac{p^{n+1} - 1}{p - 1} \frac{q^{m+1} - 1}{q - 1} \frac{r^{l+1} - 1}{r - 1} = 2 p^n q^m r^l \tag{17}$$

As before when p q and r are large, the left hand side tends towards pqr so we only need to search for small values of p, q and r. Here are the first few cases where $q = 3$ and $r = 5$:

With $p = 7$, $q = 3$ and $r = 5$, $N = 105$ and $N_s = 96$ which is a deficiency of 17%
With $p = 7$, $q = 3^2$ and $r = 5$, $N = 315$ and $N_s = 312$ which is a deficiency of 2%..
With $p = 7$, $q = 3^3$ and $r = 5$, $N = 945$ and $N_s = 960$ which is an excess of 3%.

With $p = 7$, $q = 3$ and $r = 5^2$, $N = 525$ and $N_s = 496$ which is a deficiency of 11%.
With $p = 7$, $q = 3^2$ and $r = 5^2$, $N = 1575$ and $N_s = 1612$ which is an excess of 5%.
With $p = 7$, $q = 3^3$ and $r = 5^2$, $N = 4725$ and $N_s = 4960$ which is an excess of 10%.

With $p = 11$, $q = 3$ and $r = 5^2$, $N = 825$ and $N_s = 744$ which is a deficiency of 20%.
With $p = 11$, $q = 3^2$ and $r = 5^2$, $N = 2475$ and $N_s = 2418$ which is a deficiency of 5%.
With $p = 11$, $q = 3^3$ and $r = 5^2$, $N = 7425$ and $N_s = 7440$ which is an excess of 0.4%.

In each of these sequences we move from deficiency to excess without finding a perfect number.

When we try numbers of the form $p^n q^m \times 5^2$ things look more promising and we come pretty close when $N = 3^5 \times 13 \times 5^2 = 78975$. The sum of the factors of this number is 78988 – only 13 too many!

It is clear that there are a large number of possibilities which can be serialised in many different ways. In some cases you can prove that the deficiency or excess will converge on a limit, but in those cases where a deficiency turns into an excess, only trial and error will produce a definitive answer.

Now, as is well known, no odd perfect number has ever been found – but my researches have indicated that it is not impossible for one to exist. On the other hand, it is also known that if an odd perfect number exists it must have at least 10 different primes so I think I will draw my own researches to a close here!

Carr Bank

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